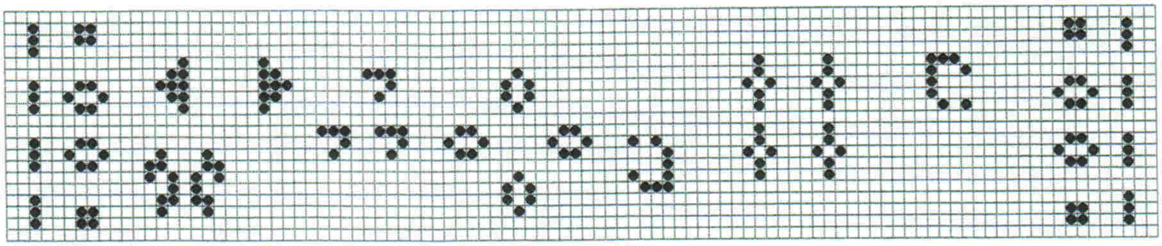


Great things are not done by impulse, but by a series of small things brought together. – Vincent van Gogh

CHAPTER 21 Sequences & Series

A **sequence** is simply a list of numbers, such as

$$2, 4, 6, 8, 10.$$

The sequence above is called a **finite sequence** because it has a finite number of terms. In other words, the sequence ends. Not all sequences end. We use "..." to indicate that a sequence continues forever:

$$2, 4, 6, 8, 10, \ldots$$

Some sequences have obvious patterns, such as

$$1, 2, 3, 4, 5, 6, 7, \ldots \qquad \text{and} \qquad 1, 2, 4, 8, 16, 32, \ldots.$$

In this chapter, we'll study the properties of a couple common examples of sequences that have useful patterns.

When using variables to represent a sequence, we often use the same letter with different subscripts to represent the terms. For example, we might represent a sequence with 5 terms as a_1, a_2, a_3, a_4, a_5.

When we add the terms of a sequence, we form a **series**. Some example series are:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$
$$2 + 4 + 6 + 8 + 10 + 12 + 14$$
$$100 + 99 + 98 + 97 + 96 + 95 + 94$$
$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$$

We could evaluate these series by simply performing lots and lots of addition. However, for some special types of series, there are much simpler ways to compute the sum.

21.1 Arithmetic Sequences

You probably recognize all of the sequences below, and have no trouble guessing what the next few terms are in each:

$$1, 2, 3, 4, 5, 6, 7, \ldots$$
$$2, 4, 6, 8, 10, 12, 14, \ldots$$
$$100, 99, 98, 97, 96, 95, \ldots$$

Each of these is an **arithmetic sequence**. In an arithmetic sequence, the difference between two consecutive terms is always the same. Such a regular pattern in the sequence makes arithmetic sequences relatively easy to understand and analyze.

Problems

Problem 21.1: Consider the sequence

$$-9, -5, -1, 3, 7, \ldots$$

in which each term is 4 greater than the previous term.

(a) Find the sixth, seventh, and eighth terms of the sequence.

(b) Find a formula for the n^{th} term of the sequence.

Problem 21.2:

(a) Must the fifth term of an arithmetic sequence with at least eight terms always be the average of the second and eighth terms?

(b) Suppose that x, y, and z are in an arithmetic sequence such that y is exactly between x and z in the sequence. (In other words, there are just as many terms between x and y as there are between y and z.) Must y be the average of x and z?

Problem 21.3: The sequence

$$4, x_1, x_2, x_3, x_4, 18$$

is an arithmetic sequence. In this problem, we find x_3.

(a) How many "steps" must we take to get from the first term, 4, to the last term, 18?

(b) Use your answer to (a) to determine how large each such step is, and use this to determine x_3.

Problem 21.4: The sum of the second term and the ninth term of an arithmetic sequence is −4. The sum of the third and fourth terms of the same sequence is 4. In this problem, we find the first term of the sequence.

(a) Let the first term be a and the common difference between terms be d, so that the second term is $a + d$. In terms of a and d, what are the third, fourth, and ninth terms?

(b) Use your answers to part (a) to solve the problem.

The pattern in an arithmetic sequence is so simple that we can easily find a formula for all the numbers in the sequence.

Problem 21.1: Consider the sequence

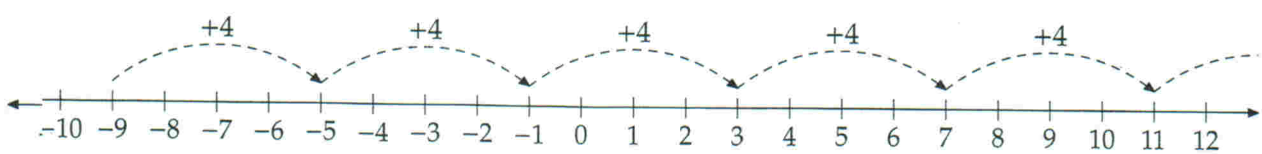
$$-9, -5, -1, 3, 7, \ldots,$$

in which each term is 4 larger than the previous term. Find a formula for the n^{th} term in the sequence.

Solution for Problem 21.1: The pattern in the sequence is clear, and we can easily generate as many more terms as we like:

$$-9, -5, -1, 3, 7, 11, 15, 19, 23, 27, 31, \ldots$$

However, it would be much easier to generate later terms in the sequence if we simply had a formula rather than having to generate a huge list. We can find these later terms by considering how many steps they are from the first term. For example, to get to the 8^{th} term, we start from the first term and take 7 steps. The diagram below illustrates our forming the sequence by taking steps of 4.



Rather than adding 4 seven times, we simply note that taking 7 steps of size 4 means adding $7 \times 4 = 28$ to our first term, so the 8^{th} term is $-9 + 28 = 19$.

Similarly, to get the n^{th} term, we start from the first term and take $n - 1$ rightward steps of size 4 steps. (Make sure you see why it is $n - 1$ steps, not n steps.) Since taking $n - 1$ rightward steps of size 4 means adding $4(n - 1)$ to the first term, this takes us to the number

$$-9 + 4(n - 1).$$

This gives us our formula for the **general term** of the sequence. We can plug in $n = 1, 2, 3, 4$, etc., to produce the sequence. Check it and see! □

We call the size of the "steps" between terms in an arithmetic sequence the **common difference** of the sequence. In exactly the same manner as in the previous problem, we can generate a formula for the general term of any arithmetic sequence given its first term and its common difference.

Important: The n^{th} term of an arithmetic sequence that has first term a and common difference d is

$$a + (n - 1)d.$$

We can use this formula to solve nearly any arithmetic sequence problem. However, a simple understanding of arithmetic sequences usually leads to an even faster solution.

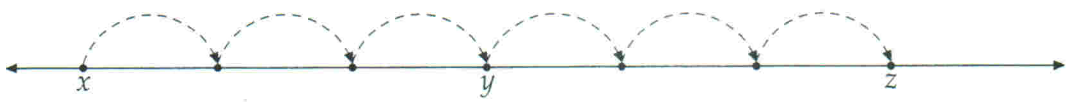
Let's take a look at one property of arithmetic sequences that often simplifies arithmetic sequence problems.

Problem 21.2: Suppose that x, y, and z are in an arithmetic sequence such that y is directly between x and z in the sequence. (In other words, there are just as many terms between x and y as there are between y and z.) Must y be the average of x and z?

Solution for Problem 21.2: To get a feel for the problem, we try a specific case.

Concept: Experimenting with special cases is a great way to develop an understanding of a general statement.

Suppose x is the 2^{nd} term, y is the 5^{th} term, and z is the 8^{th} term, so that y is directly between x and z in the sequence.



Intuitively, it is clear that y is the average of x and z, since x is three steps before y and z is three steps after y. We can prove that y is the average of x and z by letting d be our common difference and noting that

$$x = y - 3d,$$
$$z = y + 3d.$$

Adding these equations gives $x + z = 2y$, so $(x + z)/2 = y$, as desired.

This example gives us a clear path to proving that if y is directly between x and z in an arithmetic sequence, then y is the average of x and z. Let d be the common difference and k be the number of steps between x and y. The number of steps between y and z therefore is also k, so we have

$$x = y - kd,$$
$$z = y + kd.$$

Adding these equations gives $x + z = 2y$, so $(x + z)/2 = y$. Therefore, y is the average of x and z. □

This problem suggests why the average of a group of numbers is also sometimes called the **arithmetic mean** of the numbers.

Try to solve the following problem first by using the formula we developed earlier, then again using your understanding of arithmetic sequences (in other words, without using the formula).

Problem 21.3: The sequence

$$4, x_1, x_2, x_3, x_4, 18$$

is an arithmetic sequence. Find x_3.

Solution for Problem 21.3: We offer a "formula" solution and an "intuitive" solution.

Solution 1: Use the formula. We are given the first term, so we already know $a = 4$. We are also given the sixth term. Letting the common difference between terms be d, our sixth term gives us the equation

$$4 + (6 - 1)d = 18.$$

Solving this equation gives $d = \frac{14}{5}$. We must find x_3, which is the fourth term in the sequence. Our formula gives

$$x_3 = 4 + (4 - 1)d = 4 + 3\left(\frac{14}{5}\right) = \frac{62}{5}.$$

Solution 2: Use our understanding of arithmetic sequences. The 18 at the end of the sequence is 5 steps from 4. These 5 steps cover a distance of $18 - 4 = 14$, so each step has length $\frac{14}{5}$. We must take three such steps to get from the first term to x_3, so

$$x_3 = 4 + 3\left(\frac{14}{5}\right) = \frac{62}{5}.$$

Notice that our solutions are essentially the same. □

WARNING!! Don't simply memorize the formulas in this chapter. If you take the time to understand them, you'll be able to solve problems much more quickly with much less likelihood of making a mistake. Also, once you understand the formulas, you won't have to memorize them – you'll simply know them.

And once you do know them, you'll have no difficulty tackling problems like the following one.

Problem 21.4: The sum of the second term and the ninth term of an arithmetic sequence is -4. The sum of the third and fourth terms of the same sequence is 4. Find the first term of the sequence.

Solution for Problem 21.4: Letting the first term be a and the common difference be d, we have

$$\begin{aligned} \text{Second term} &= a + d, \\ \text{Ninth term} &= a + 8d, \\ \text{Third term} &= a + 2d, \\ \text{Fourth term} &= a + 3d. \end{aligned}$$

Using these expressions with the given information about sums of these terms, we have the equations

$$\begin{aligned} (a + d) + (a + 8d) &= -4, \\ (a + 2d) + (a + 3d) &= 4. \end{aligned}$$

Simplifying the left hand sides gives the equations

$$\begin{aligned} 2a + 9d &= -4, \\ 2a + 5d &= 4. \end{aligned}$$

Subtracting the second equation from the first gives $4d = -8$, so $d = -2$. Substituting this into either of our equations gives $a = 7$, so the first term of the sequence is 7. □

Exercises

21.1.1 Consider the arithmetic sequence $1, 4, 7, 10, 13, \ldots$

(a) Find the 15^{th} term in the sequence.

(b) Find a formula for the n^{th} term in the sequence.

21.1.2 The third term of an arithmetic sequence is 5 and the sixth term is -1. Find the twelfth term of this sequence.

21.1.3 How many terms are in the arithmetic sequence $5, 11, 17, \ldots, 89$?

21.1.4 When the 171^{st} even positive integer is subtracted from the 219^{th} odd positive integer, the result is z. Find z. *(Source: MATHCOUNTS)*

21.1.5★ In the infinite arithmetic sequence $a_1, a_2, a_3, \ldots$, we have $a_8 = 2001$. If the common difference d is an integer, find the minimum value of d so that $a_{17} > 10000$. **Hints:** 5

21.2 Arithmetic Series

When we add a group of consecutive terms of an arithmetic sequence, we form an **arithmetic series**. For example, the series

$$1 + 2 + 3 + 4 + \cdots + 99 + 100$$

is an arithmetic series.

Problems

Problem 21.5: According to legend, the great mathematician Carl Gauss was given the busy-work assignment in elementary school of finding the sum of the first 100 positive integers. (Busy work is not a recent invention – Gauss was born in 1777.) While the other students scribbled away tediously adding and adding and adding, Gauss thought briefly, then wrote down the correct answer. In this problem, we re-create the method he allegedly used to find the sum:

$$1 + 2 + 3 + 4 + \cdots + 99 + 100.$$

(a) Write the sum backwards, starting with 100 and ending at 1.

(b) Add the series you wrote in part (a) to the original series by summing the first terms of each series, then summing the second terms of each series, then summing the third terms, and so on. Do you see anything interesting in your sums?

(c) Find the sum $1 + 2 + 3 + \cdots + 99 + 100$.

Problem 21.6: Suppose we have an arithmetic series with first term a, common difference d, and with n terms. Use the previous problem as inspiration to prove the following:

(a) The arithmetic series has sum
$$(\text{Number of terms}) \cdot \frac{\text{First term} + \text{Last term}}{2}.$$

(b) The arithmetic series has sum
$$\frac{n[2a + (n-1)d]}{2}.$$

Problem 21.7: Evaluate the following arithmetic series:

(a) $-3 - 1 + 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$.

(b) $73 + 67 + 61 + \cdots + 7$.

(c) $\frac{7}{6} + \frac{4}{3} + \frac{3}{2} + \cdots + 11$.

Problem 21.8:

(a) Find a formula for the sum of the first n positive integers.

(b) For how many positive integers n does $1 + 2 + \cdots + n$ evenly divide $6n$? *(Source: AMC 12)*

(c) Find a formula for the sum of the first n positive odd integers. (If you can't find the formula right away, find the sum for a few n first and look for a pattern.)

Problem 21.9: The sum of the first nine terms of an arithmetic sequence is 72. What is the fifth term of the sequence?

Problem 21.10: Let $a_1, a_2, \ldots, a_k$ be an arithmetic sequence with
$$a_4 + a_7 + a_{10} = 17$$
and
$$a_4 + a_5 + a_6 + \cdots + a_{12} + a_{13} + a_{14} = 77.$$
In this problem we find the value of k for which $a_k = 13$. *(Source: AHSME)*

(a) *Solution 1: Use our formulas.* Let a_1 be the first term and d the common difference. Use the given equations to find a_1, d, and then k.

(b) *Solution 2: Use your understanding of arithmetic sequences.* Use the first equation to find a_7. Use the second to find a_9. Then find the common difference and the desired value of k.

Problem 21.5: Find the sum $1 + 2 + 3 + \cdots + 99 + 100$.

Solution for Problem 21.5: We could find the sum by hand ... eventually, and we'd probably make an error. Instead, we have to find a more clever approach. Looking at the few terms we have written, we see an interesting relationship. The sum of the first and last terms is

$$1 + 100 = 101.$$

The sum of the second and next-to-last terms is

$$2 + 99 = 101.$$

Continuing, we see that we can pair off terms such that each pair has sum 101. One way we can use this observation is to add the series to itself. We let the sum equal S. We write the sum forwards and then again backwards, so that when we add, we pair up terms that add to 101:

$$\begin{array}{rcrcrcrcrcccrcr} S & = & 1 & + & 2 & + & 3 & + & 4 & + & \cdots & + & 99 & + & 100 \\ +S & = & 100 & + & 99 & + & 98 & + & 97 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & 101 & + & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

There are clearly 100 terms in the sum on the right, because we are adding 100 numbers. Therefore, we have $2S = 101(100)$, so $S = 101(50) = 5050$. □

Concept: Some series can be evaluated by combining two copies of the series in a clever way.

With this example under our belts, we're ready to prove some general formulas for arithmetic series.

Sidenote: The plural of series is series.

Problem 21.6: Suppose we have an arithmetic series with first term a, common difference d, and with n terms. Prove the following:

(a) The arithmetic series has sum

$$(\text{Number of terms}) \cdot \frac{\text{First term} + \text{Last term}}{2}.$$

(b) The arithmetic series has sum

$$\frac{n[2a + (n-1)d]}{2}.$$

Solution for Problem 21.6: Our solution to the previous problem provides a guide. We saw there that we can pair off terms such that each pair has sum equal to

$$\text{First term} + \text{Last term}.$$

Since there are n total terms, there are $n/2$ pairs of terms, where n is the number of terms in the series. Therefore, the sum of all the terms is

$$\frac{\text{Number of terms}}{2} \cdot (\text{First term} + \text{Last term}).$$

Notice that we can also write this as

$$(\text{Number of terms}) \cdot \frac{\text{First term} + \text{Last term}}{2},$$

or

$$(\text{Number of terms}) \times (\text{Average of first and last term}).$$

This explanation is not yet a proof. First, what if there are an odd number of terms? (Hint: What term equals the average of the first and last terms in an arithmetic sequence with an odd number of terms?) Second, we haven't proved that we can pair off all the terms into pairs with the same sum.

We take care of both these complaints by looking at our general form for an arithmetic series. We let S be our sum, a be the first term, d be the common difference, and n be the number of terms. As before, we write our series forwards and backwards:

$$\begin{array}{rcccccccccc}
S & = & a & + & (a+d) & + & \cdots & + & [a+(n-2)d] & + & [a+(n-1)d] \\
+S & = & [a+(n-1)d] & + & [a+(n-2)d] & + & \cdots & + & (a+d) & + & a \\
\hline
2S & = & [2a+(n-1)d] & + & [2a+(n-1)d] & + & \cdots & + & [2a+(n-1)d] & + & [2a+(n-1)d]
\end{array}$$

In each pair of terms that we add, we are adding the term that is k steps from the beginning, $a + kd$, to the term that is k steps from the end, $a + (n - 1 - k)d$. The sum of these is always $2a + (n - 1)d$, no matter what k is. Notice that this takes care of showing that the paired off sums are all the same, and of the issue of an odd number of terms (the middle term is added to itself).

There are n terms in our series, so there are n terms on the right side of the sum equal to $2S$. Therefore, we have

$$S = \frac{n[2a + (n-1)d]}{2}.$$

Because the first term of the series is a and the last is $a + (n - 1)d$, the expression $2a + (n - 1)d$ equals the sum of the first and last terms. So, we have

$$S = \frac{n[2a + (n-1)d]}{2} = (\text{Number of terms}) \cdot \frac{\text{First term} + \text{Last term}}{2}.$$

□

Important: The sum of an arithmetic series equals

$$(\text{Number of terms}) \times (\text{Average of first and last term}).$$

If the first term is a, the common difference is d, and the number of terms is n, we can write this as

$$\frac{n[2a + (n-1)d]}{2}.$$

Don't memorize that last formula! If you understand that the sum of an arithmetic series is the product of the number of terms in the series and the average of the first and last terms, you'll be able to reproduce the formula quickly whenever you need it.

Problem 21.7: Evaluate the following arithmetic series:

(a) $-3-1+1+3+5+7+9+11+13+15$.

(b) $73+67+61+\cdots+7$.

(c) $\frac{7}{6}+\frac{4}{3}+\frac{3}{2}+\cdots+11$.

Solution for Problem 21.7:

(a) There are 10 terms. The first term is -3 and the last 15, so the sum is

$$\frac{-3+15}{2}\cdot 10 = 60.$$

(b) We have the first and last term, but we need to figure out the number of terms. The common difference is -6. In going from 73 down to 7, we decrease by $73-7=66$, which is $66/6=11$ steps of 6 downward. Therefore, there are 12 terms. (Alternatively, we could solve the equation $73-6k=7$ to determine how many steps we must take to get from the first term to the last term.) So, our sum is

$$\frac{73+7}{2}\cdot 12 = 480.$$

(c) The common difference is $\frac{4}{3}-\frac{7}{6}=\frac{1}{6}$. Letting there be n terms in the series, we have

$$\frac{7}{6}+(n-1)\left(\frac{1}{6}\right)=11.$$

Make sure you see why we use $n-1$ in this equation, not n: we take $n-1$ steps to go from the first term to the n^{th} term. Solving this equation gives $n=60$, so our sum is

$$\frac{\frac{7}{6}+11}{2}\cdot 60=\left(\frac{7}{6}+11\right)\cdot\frac{60}{2}=\left(\frac{7}{6}+11\right)(30)=35+330=365.$$

□

Problem 21.8:

(a) Find a formula for the sum of the first n positive integers.

(b) For how many positive integers n does $1+2+\cdots+n$ evenly divide $6n$? *(Source: AMC 12)*

(c) Find a formula for the sum of the first n positive odd integers.

Solution for Problem 21.8:

(a) We seek a formula for the sum

$$1+2+3+4+\cdots+(n-1)+n.$$

The first term is 1, the last term is n, and there are n terms, so our sum is

$$\frac{n+1}{2}\cdot n=\frac{n(n+1)}{2}.$$

The sum of the first n positive integers is so common that it's worth knowing and understanding this sum well.

Important: The sum of the first n positive integers is

$$\frac{n(n+1)}{2}.$$

(b) The sum of the first n positive integers is $n(n+1)/2$, so we seek the number of values of n for which

$$\frac{6n}{\frac{n(n+1)}{2}} = \frac{12n}{n(n+1)} = \frac{12}{n+1}$$

is an integer (where $n > 0$). Since $n+1$ divides 12 only for $n = 1, 2, 3, 5$, and 11, there are 5 values of n such that the sum of the first n positive integers divides $6n$.

(c) Before we find the general sum, let's explore a little bit:

$$\begin{aligned} 1 &= 1, \\ 1+3 &= 4, \\ 1+3+5 &= 9, \\ 1+3+5+7 &= 16. \end{aligned}$$

It sure looks like the sum of the first n positive odd integers is n^2. To prove it, we first find the n^{th} odd integer. The positive odd integers form an arithmetic sequence with first term 1 and common difference 2, so the n^{th} positive odd integer is $1 + 2(n-1) = 2n - 1$. Therefore, the sum of the first n positive odd integers is

$$\frac{1 + (2n-1)}{2} \cdot n = \frac{2n}{2} \cdot n = n^2,$$

as expected.

□

Important: The sum of the first n positive odd integers is n^2.

Now that we know how to find the sum of a basic arithmetic series, let's apply our knowledge to some more interesting problems.

Problem 21.9: The sum of the first nine terms of an arithmetic sequence is 72. What is the fifth term of the sequence?

Solution for Problem 21.9: We offer two solutions:

Solution 1: Let a be the first term of the sequence. And, as usual, we let d be the common difference. Since there are 9 terms, the first of which is a and the last of which is $a + 8d$, we have

$$\frac{a + (a+8d)}{2} \cdot 9 = 72.$$

Therefore, we have $a + 4d = 8$. We can't directly find a or d from this equation. However, we aren't looking for a or d; we're looking for the fifth term.

> **Concept:** Keep your eye on the ball. If the problem asks for something more complicated than the value of a specific variable, write an algebraic expression for what is sought.

The fifth term is $a + 4d$, which we already know is equal to 8.

Solution 2: Let a be the middle term of the sequence. We know that the terms in an arithmetic sequence can be paired so that each pair has an average equal to this middle term. So, we let a be the middle term and we let d be the common difference. So, our sequence is

$$a - 4d, a - 3d, a - 2d, a - d, a, a + d, a + 2d, a + 3d, a + 4d.$$

Now we see the power of letting a be the middle term. We can easily sum these 9 terms because all the d's cancel, leaving $9a$ as the sum. We are given that this equals 72, so $a = 8$. The middle term is our fifth term, so the desired fifth term is 8. □

Our second solution gives us another clever way to think about arithmetic series.

> **Concept:** Many arithmetic series problems can be tackled by considering the terms relative to one of the middle terms, rather than as a number of steps from the beginning term. This tactic is particularly useful when working with arithmetic series with an odd number of terms, because the sum of such a series equals the middle term times the number of terms.

Problem 21.10: Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17$$

and

$$a_4 + a_5 + a_6 + \cdots + a_{12} + a_{13} + a_{14} = 77.$$

For what value of k does $a_k = 13$?

Solution for Problem 21.10: We offer two solutions:

Solution 1: Pound away with the formulas. We can solve this problem by using our formulas blindly. The first term of the sequence is a_1. Let the common difference be d. Since a_4 is 3 terms after a_1, we have $a_1 = a_4 + 3d$. Similarly, $a_7 = a_1 + 6d$ and $a_{10} = a_1 + 9d$. Our first equation then becomes

$$(a_1 + 3d) + (a_1 + 6d) + (a_1 + 9d) = 17.$$

Simplifying this equation gives $3a_1 + 18d = 17$.

We can also write all the terms in our second equation in terms of a_1 and d, and we have

$$(a_1 + 3d) + (a_1 + 4d) + \cdots + (a_1 + 12d) + (a_1 + 13d) = 77.$$

There are 11 terms in the series $a_4 + a_5 + \cdots + a_{14}$, so we have 11 a_1 terms on the left. Furthermore, adding the coefficients of the d terms on the left gives us

$$\frac{3 + 13}{2} \cdot 11 = 88,$$

so our second equation is now $11a_1 + 88d = 77$. Dividing this equation by 11 gives $a_1 + 8d = 7$.

We therefore have the system of equations

$$\begin{aligned} 3a_1 + 18d &= 17, \\ a_1 + 8d &= 7. \end{aligned}$$

We can eliminate a_1 by subtracting 3 times the second equation from the first, which gives $-6d = -4$, or $d = 2/3$. Substituting this into either equation above gives $a_1 = 5/3$. Therefore, our arithmetic sequence starts with $5/3$ and increases by steps of size $2/3$. The term a_k is $k - 1$ steps after the first term, so the value of k such that $a_k = 13$ is the solution to

$$\frac{5}{3} + \frac{2}{3}(k-1) = 13.$$

The solution to this equation is $k = 18$.

Concept: We can often solve problems involving arithmetic sequences or series by writing equations involving the first term, the common difference, and/or the number of terms.

Solution 2: Use our understanding of arithmetic sequences. We start with the equation $a_4 + a_7 + a_{10} = 17$. Letting d be the common difference, we note that $a_4 = a_7 - 3d$ and $a_{10} = a_7 + 3d$, so $a_4 + a_7 + a_{10} = 3a_7$. This makes the equation $3a_7 = 17$, so $a_7 = 17/3$.

Similarly, we can write the series in the second equation,

$$a_4 + a_5 + a_6 + \cdots + a_{12} + a_{13} + a_{14} = 77,$$

in terms of its middle term, a_9. Since there are 11 terms in this series, and their average is a_9, the series equals $11a_9$, giving us the equation $11a_9 = 77$, or $a_9 = 7$.

Since our sequence increases by $7 - \frac{17}{3} = \frac{4}{3}$ in taking the 2 steps from a_7 to a_9, the common difference is $(4/3)/2 = 2/3$. From here we can find the desired k in many ways. One clever way to do it is to note that for every 3 steps, the terms in the series increase by 2. To get from $a_9 = 7$ to $a_k = 13$, we need to increase by 2 three times. We therefore must take 3 steps three times, for a total of 9 steps past a_9. This brings us to $a_{18} = 13$, so $k = 18$, as before. □

Exercises

21.2.1 Compute the sum of each of the following arithmetic series:

(a) $21 + 28 + 35 + \cdots + 105$

(b) The arithmetic series with first term 7, common difference -3, and 14 terms

(c) $\frac{1}{2} + \frac{5}{6} + \frac{7}{6} + \cdots + \frac{19}{2}$

21.2.2 The sum of a 15-term arithmetic series with first term 7 is -210. What is the common difference?

21.2.3 The sum of the first 5 terms of an arithmetic series is 70. The sum of the first 10 terms of this arithmetic series is 210. What is the first term of the series?

21.2.4 Explain why an arithmetic series with an odd number of terms has its sum equal to the number of terms times the middle term of the series.

21.2.5 The sum of 5 consecutive even integers is 4 less than the sum of the first 8 consecutive odd positive integers. What is the smallest of the even integers? *(Source: AMC 10)*

21.2.6 If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, then what is the sum of the first $4n$ positive integers?

21.2.7 Suppose that the sequence $a_1, a_2, a_3, \ldots, a_{200}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $a_2 - a_1$? *(Source: AMC 10)*

21.2.8★ The **arithmetic mean** can be extended to more than just two numbers. The arithmetic mean of the numbers $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(a) Suppose $a_1 \le a_2 \le \cdots \le a_n$. Why must the arithmetic mean of the numbers $a_1, a_2, \ldots, a_n$ be at least a_1, but no greater than a_n?

(b) Suppose $a_1, a_2, \ldots, a_n$ is an arithmetic sequence. Show that the arithmetic mean of all the terms in the sequence is the same as the arithmetic mean of a_1 and a_n.

21.3 Geometric Sequences

Back on page 525 of Section 19.1 we introduced the legend in which a clever girl, Meena, requested a special reward from the king. Meena asked that the king place a single grain of rice on the first square of her chessboard. She asked that on each day thereafter, the king place on the next square of her chessboard twice the amount of rice he had placed on the board the day before.

The number of grains the king must give Meena on successive days form a **geometric sequence**:

$$1, 2, 4, 8, 16, 32, \ldots.$$

In a geometric sequence, instead of there being a common difference between terms, there is a common ratio between terms. That is, after the first term, the ratio between each term and the preceding term is always the same.

Problems

Problem 21.11: Consider the sequence $1.25, 2.5, 5, 10, 20, \ldots$, in which each term is double the term before it.

(a) Find the sixth, seventh, and eighth terms in the sequence.

(b) Find an expression for the n^{th} term in the sequence.